JOURNAL OF APPROXIMATION THEORY 9, 395–397 (1973)

A Characterization of Functions Having Zygmund's Property

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1. Let $-\infty < a < b < \infty$. Given a real function f, we say that it has *Zygmund's property* in $[a, b]$ iff it is continuous there, and for some constant A,

$$|f(x) - 2f(x+h) + f(x+2h)| \leqslant Ah \tag{1}$$

whenever $a \leqslant x < x + 2h \leqslant b$.

By using the method of proof of Theorem (3.4) of [1], one can show that "continuous" in the last sentence can be replaced by "defined and bounded."

If $b - a > 2\pi$, if f is a 2π-periodic real function on $(-\infty, \infty)$, and if it has Zygmund's property in $[a, b]$ with a constant A, then (1), with some constant A' replacing A, holds whenever $-\infty < x < x + 2h < \infty$; further, by a fundamental result of Zygmund [2, Theorem 8], for $n = 1, 2,\ldots$ there is a $t_n \in T_n$ for which

$$\sup_{-\infty<x<\infty} |f(x) - t_n(x)| \leqslant B/n, \tag{2}$$

B being a constant, where T_n is the set of all trigonometric polynomials of the form $a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$, with real a's and b's.

Conversely, if for $n = 1, 2,\ldots$ there is a $t_n \in T_n$ satisfying (2) for some constant B and some real function f on $(-\infty, \infty)$, then, by Zygmund's result, (1) holds whenever $-\infty < x < x + 2h < \infty$, A being some constant.

The purpose of the present note is to characterize functions having Zygmund's property in $[a, b]$ by means of a sequence L_n of families of extremely simple functions which, like trigonometric polynomials, are very natural in the context of contemporary approximation theory.

2. For $n = 1, 2,\ldots$ set $x_k^{(n)} = a + k(b-a)\,n^{-1}$, $k = 0, 1,\ldots, n$, so that $(x_k^{(n)})_{k=0}^{n}$ partitions $[a, b]$ into n congruent subintervals $[x_{k-1}^{(n)}, x_k^{(n)}]$, $k = 1, 2,\ldots, n$. Also, for $n = 1, 2,\ldots$ let L_n be the set of all real functions with domain $[a, b]$ which are linear in each such subinterval.

THEOREM. *Given a real function f on $[a, b]$, a necessary and sufficient* *condition for it to have Zygmund's property in* $[a, b]$ *is that for* $n = 1, 2, \ldots$ *there exists an* $\ell_n \in L_n$ *such that*

$$\sup_{a \leqslant x \leqslant b} |f(x) - \ell_n(x)| \leqslant C/n, \tag{3}$$

C being a constant.

3. LEMMA. *Let f^* be a real function, continuous in* $[0, 1]$, *and let* ℓ^* *be the linear function with domain* $[0, 1]$ *satisfying* $\ell^*(0) = f^*(0)$, $\ell^*(1) = f^*(1)$. *Suppose, for some constant A,* (1) (*with f^* replacing f*) *holds whenever* $0 \leqslant x < x + 2h \leqslant 1$. *Then*

$$|f^*(x) - \ell^*(x)| \leqslant A/2 \qquad \textit{throughout } [0, 1]. \tag{4}$$

Proof of the Lemma. We assume, as we may, that $\ell^*(x) \equiv 0$ (otherwise, replace f^* by $f^*(x) + [f^*(0) - f^*(1)]x - f^*(0)$). We shall show that if $0 \leqslant y \leqslant 1$, $y = \sum_{k=0}^{n} a_k 2^{-k}$, $n \geqslant 0$, where each a_k is 0 or 1, then

$$|f^*(y)| \leqslant (A/2)(1 - 2^{-n}). \tag{5}$$

This will imply (4), since if $0 \leqslant x \leqslant 1$, $x = \sum_{k=0}^{\infty} a_k 2^{-k}$, where each a_k is 0 or 1, then $|f^*(x)| = \lim_{n\to\infty} |f^*(\sum_{k=0}^{n} a_k 2^{-k})| \leqslant A/2$.

Now (5) obviously holds for $n = 0$. Suppose it holds for some $n \geqslant 0$. Let $0 \leqslant y \leqslant 1$, $y = \sum_{k=0}^{n+1} a_k 2^{-k}$, where each a_k is 0 or 1. We shall prove

$$|f^*(y)| \leqslant (A/2)(1 - 2^{-n-1}). \tag{6}$$

We can obviously assume, by the induction hypothesis, that $a_{n+1} = 1$. Set

$$u = y - 2^{-n-1} = \sum_{k=0}^{n} a_k 2^{-k}, \qquad v = y + 2^{-n-1}.$$

One easily sees that $0 < v \leqslant 1$, $v = \sum_{k=0}^{n} a_k' 2^{-k}$, where each a_k' is 0 or 1. Thus

$$|f^*(u)| \leqslant (A/2)(1 - 2^{-n}), \tag{7}$$

$$|f^*(v)| \leqslant (A/2)(1 - 2^{-n}). \tag{8}$$

Since

$$2f^*(y) = f^*(u) + f^*(v) - [f^*(u) - 2f^*(u + 2^{-n-1}) + f^*(u + 2.2^{-n-1})],$$

(6) follows from (7), (8), and (1).

4. *Proof of the Theorem.* Suppose f has Zygmund's property in $[a, b]$. Let n be a positive integer, and let $\ell_n \in L_n$ be defined by the requirement $\ell_n(x_k^{(n)}) = f(x_k^{(n)})$, $k = 0, 1,..., n$. We shall prove:

$$|f(t) - \ell_n(t)| \leqslant (A/2)(b - a)/n \tag{9}$$

for every $t \in [a, b]$. Consider such a t, and let $x_{j-1}^{(n)} \leqslant t \leqslant x_j^{(n)}$, $1 \leqslant j \leqslant n$. Define f^*, $\ell_n{}^*$, with domain $[0, 1]$, as follows:

$$f^*(x) \equiv f(x_{j-1}^{(n)} + x(b - a)\, n^{-1}), \qquad \ell_n{}^*(x) \equiv \ell_n(x_{j-1}^{(n)} + x(b - a)\, n^{-1}).$$

Then $|f^*(x) - 2f^*(x + h) + f^*(x + 2h)| \leqslant A(b - a)\, n^{-1}h$ whenever $0 \leqslant x < x + 2h \leqslant 1$. Hence, by the lemma, $|f^*(x) - \ell_n{}^*(x)| \leqslant (A/2)(b - a)/n$ throughout $[0, 1]$. In particular, taking $x = (t - a)\, n(b - a)^{-1} - (j - 1)$, we obtain (9).

Conversely, suppose for $n = 1, 2,...$ there exists an $\ell_n \in L_n$ satisfying (3). Clearly f is continuous in $[a, b]$; we shall show that (1) holds whenever $a \leqslant x < x + 2h \leqslant b$, with $A = 48C(b - a)^{-1}$.

Let $a \leqslant x < x + 2h \leqslant b$. Let n_0 be the largest positive integer n for which $[x, x + 2h]$ lies in some $[x_{k-1}^{(n)}, x_k^{(n)}]$, $1 \leqslant k \leqslant n$. Then $2h > (b - a)(6n_0)^{-1}$. For otherwise, if, say $[x, x + 2h] \subseteq I = [x_{k_0-1}^{(n_0)}, x_{k_0}^{(n_0)}]$, $1 \leqslant k_0 \leqslant n_0$, then $[x, x + 2h]$ would lie either in one of the two (closed) halves of I or in the (open) middle third of I. In each case, the maximality of n_0 is contradicted.

Using the linearity of ℓ_{n_0} in $[x, x + 2h]$, we have:

$$\begin{aligned} &|f(x) - 2f(x + h) + f(x + 2h)| \\ &\quad = |\{f(x) - \ell_{n_0}(x)\} - 2\{f(x + h) - \ell_{n_0}(x + h)\} \\ &\qquad + \{f(x + 2h) - \ell_{n_0}(x + 2h)\}| \\ &\quad \leqslant 4C/n_0 \leqslant 48C(b - a)^{-1}h. \end{aligned}$$

The necessity of the condition of the theorem follows also from Zusatz 1.2 of [3] and from Theorem 2 of [4].

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